

Homework 12

Math 121A, Fall 2025.

Due: Wednesday, Nov 19, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. (*Fourier Transform Diagonalizes Translations*)

Fix any $h \in \mathbb{R}$. Similarly to last homework, the following “translation” operator $\tau_h : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ given by

$$(\tau_h f)(x) = f(x - h), \quad x \in \mathbb{R}$$

is unitary (you do not need to prove this).

- (a) Show the following two equalities. Observe that the first one just says “the Fourier transform diagonalizes translations”.

$$\begin{aligned} \widehat{\tau_h f}(k) &= e^{-ikh} \hat{f}(k) \\ \widehat{e^{ihx} f}(k) &= (\tau_h \hat{f})(k). \end{aligned}$$

- (b) Use the limit definition of the derivative to rewrite the derivative operator $\frac{d}{dx}$ in terms of the translation operator.
- (c) Suppose that we may commute limits of operators with the Fourier transform $\mathcal{F}$, meaning if A_h is a collection of operators such that $\lim_{h \rightarrow 0} A_h = A$, then $\lim_{h \rightarrow 0} \mathcal{F}(A_h) = \mathcal{F}(A)$. Use the previous parts to give an alternative proof that the Fourier transform diagonalizes derivatives, i.e.

$$\widehat{\frac{d}{dx} f}(k) = ik \hat{f}(k).$$

Problem 2. (*Frequency modulation*) Let $a \in \mathbb{R}$ be nonzero, and define $g(x) = f(ax)$. Show that

$$\hat{g}(k) = \frac{1}{|a|} \hat{f}\left(\frac{k}{a}\right).$$

Problem 3. (*Fourier Transform of the Triangle Function*)

Find the exponential Fourier transform of

$$T(x) = \begin{cases} 2a - |x|, & |x| < 2a, \\ 0, & |x| > 2a, \end{cases}$$

Note: for upcoming problems, it is useful to note that $1 - \cos(2\kappa) = 2 \sin^2(\kappa)$.

Problem 4. (*Poisson's Summation Formula*)

Define a function $h(x) = \sum_{k=-\infty}^{\infty} f(x+2k\pi)$, assuming that the series converges to a function satisfying Dirichlet conditions (Section 6 of Boas). Verify that $h(x)$ does have period 2π .

(a) Expand $h(x)$ in an exponential Fourier series

$$h(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx};$$

show that $c_n = g(n)$ where $g(\alpha)$ is the Fourier transform of $f(x)$. *Hint: Write c_n as an integral from 0 to 2π and make the change of variable $u = x + 2k\pi$. Note that $e^{-2ink\pi} = 1$, and the sum on k gives a single integral from $-\infty$ to ∞ .*

(b) Let $x = 0$ in (a) to get *Poisson's summation formula*

$$\sum_{k=-\infty}^{\infty} f(2k\pi) = \sum_{n=-\infty}^{\infty} g(n).$$

This result has many applications; for example: statistical mechanics, communication theory, theory of optical instruments, scattering of light in a liquid, and so on.

Problem 5. (*A problem from optics*)

Use Poisson's summation formula from the previous problem and the Fourier transform of the Triangle function to show that

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2 n\theta}{n^2} = \pi\theta, \quad 0 < \theta < \pi.$$

(This sum is needed in the theory of scattering of light in a liquid.) *Hint: Consider $f(x)$ and $g(\alpha)$ as in the triangle function Fourier Transform. Note that $f(2k\pi) = 0$ except for $k = 0$ if $a < \pi$. Put $\alpha = n$, $a = \theta$.*

Problem 6. (*An exponentially decaying interaction*)

Find the exponential Fourier transform of $f(x) = e^{-|x|}$ and write the inverse transform. You should find

$$\int_0^{\infty} \frac{\cos \alpha x}{\alpha^2 + 1} d\alpha = \frac{\pi}{2} e^{-|x|}.$$

Problem 7. Many Fourier transforms are best computed by using known Fourier transforms and properties of the Fourier transform. Compute the exponential Fourier transform of

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

using a known Fourier transform from class.

Problem 8. Compute the convolution of the “rectangular pulse” from class

$$f(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

with itself in two ways:

- (a) Directly compute $f \star f$.
- (b) Take Fourier transforms, multiply, then take inverse Fourier transform. (*Hint: you may want to do Problem 3 first.*)

Problem 9. (*Fourier transform of a Gaussian is a Gaussian*) Show that the normal probability function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

has a Fourier transform which is also a (rescaled) normal probability function, i.e. $\hat{f}(k) = \sqrt{2\pi} f(k)$ for all real k (*Hint: complete the square and consider the contour integral over an infinitely long rectangular box whose top is the line $\text{Im}(z) = ik$ and whose bottom is the real axis.*)

As a side remark, convolution with a Gaussian is a very common technique for “smoothing” data, e.g. in image processing.